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Off-shell (4,4) supersymmetric sigma models with torsion as gauge theories in harmonic superspace

Evgenyi A. Ivanov

Laboratoire de Physique Théorique, ENSLAPP,
ENS Lyon, 46 Allée d'Italie, F-69364 Lyon Cedex 07, France
and
Bogoliubov Laboratory of Theoretical Physics, JINR,
141 980, Dubna near Moscow, Russian Federation

Abstract

Starting from the action of $(4,4)$ $2D$ twisted multiplets in the harmonic superspace with a double set of $SU(2)$ harmonic variables, we present its generalization which provides an off-shell description of a wide class of $(4,4)$ sigma models with torsion and non-commuting left and right complex structures. The distinguishing features of the action constructed are: (i) a nonabelian and in general nonlinear gauge invariance ensuring a correct number of physical degrees of freedom; (ii) an infinite tower of auxiliary fields. For a particular class of such models we explicitly demonstrate the non-commutativity of complex structures on the bosonic target.

1. Introduction. Remarkable target geometries of $2D$ sigma models with extended worldsheet SUSY are revealed most clearly within manifestly supersymmetric off-shell superfield formulations of these theories. For torsionless $(2,2)$ and $(4,4)$ sigma models the relevant superfield Lagrangians were found to coincide with (or to be directly related to) the fundamental objects underlying the given geometry: Kähler potential in the $(2,2)$ case [1], hyper-Kähler or quaternionic-Kähler potentials in the flat or curved $(4,4)$ cases [2 - 5]. To have superfield off-shell formulations is also highly desirable while quantizing these models and proving, e.g., their ultraviolet finiteness.

An important wide class of $2D$ supersymmetric sigma models is presented by $(2,2)$ and $(4,4)$ models with torsionful bosonic target manifolds and two independent left and right sets of complex structures (see, e.g. [6, 7]). These models and, in particular, their group manifold WZNW representatives [8] can provide non-trivial backgrounds for $4D$ superstrings (see, e.g., [9]) and be relevant to $2D$ black holes [10]. A manifestly supersymmetric formulation of $(2,2)$ models with commuting left and right complex structures in terms of chiral and twisted chiral $(2,2)$ superfields and an exhaustive discussion of their geometry have been given in [7]. For $(4,4)$ models with commuting structures there exist manifestly supersymmetric off-shell formulations in the projective, ordinary and $SU(2)\times SU(2)$ harmonic $(4,4)$ superspaces [10-12]. The appropriate superfields represent, in one or another way, the $(4,4)$ $2D$ twisted multiplet [13, 7].

Much less is known about $(2,2)$ and $(4,4)$ sigma models with non-commuting complex structures, despite the fact that most of the corresponding group manifold WZNW sigma models [8] fall into this category [10]. In particular, it is unclear how to describe them off shell in general. As was argued in Refs. [7, 14, 10], twisted $(2,2)$ and $(4,4)$ multiplets are not suitable for this purpose. It has been then suggested to make use of some other off-shell representations of $(2,2)$ [14, 15] and $(4,4)$ [14, 16] worldsheet SUSY. However, it is an open question whether the relevant actions correspond to generic sigma models of this type.

In the present letter we propose another approach to the off-shell description of general $(4,4)$ sigma models with torsion, based upon an analogy with general torsionless hyper-Kähler $(4,4)$ sigma models in $SU(2)$ harmonic superspace [2 - 4]. We start from a dual form of the general action of $(4,4)$ twisted superfields in $SU(2)\times SU(2)$ analytic harmonic superspace with two independent sets of harmonic variables [12] and construct a direct $SU(2)\times SU(2)$ harmonic analog of the hyper-Kähler $(4,4)$ action. The form of the action obtained, contrary to the torsionless case, proves to be severely constrained by the integrability conditions following from the commutativity of the left and right harmonic derivatives. While for four-dimensional bosonic manifolds the resulting action is reduced to that of twisted superfield, for manifolds of dimension $4n,\ n\geq 2$, the generic action *cannot be written only in terms of twisted superfields.* Its most characteristic features are (i) the unavoidable presence of infinite number of auxiliary fields and (ii) a nonabelain and in general nonlinear gauge symmetry which ensures the necessary number of propagating fields. These symmetry and action are harmonic analogs of the Poisson gauge symmetry and actions proposed in [17, 18]. For an interesting subclass of these actions, harmonic analogs of the Yang-Mills ones, we explicitly demonstrate that the left and right complex structures on the bosonic target *do not commute.*

2. Sigma models in SU(2)xSU(2) harmonic superspace. The $SU(2)\times SU(2)$ harmonic superspace is an extension of the standard real $(4,4)$ $2D$ superspace by two independent sets of harmonic variables $u^{\pm 1\, i}$ and $v^{\pm 1\, a}$ ($u^{1\, i}u^{-1}_i = v^{1\, a}v^{-1}_a = 1$) associated with the automorphism groups $SU(2)_L$ and $SU(2)_R$ of the left and right sectors of $(4,4)$ supersymmetry [12]. The corresponding analytic subspace is spanned by the following set of coordinates

$$(\zeta, u, v) = (\; x^{++}, x^{--}, \theta^{1,0\,\underline{i}}, \theta^{0,1\,\underline{a}}, u^{\pm 1\, i}, v^{\pm 1\, a}\;)\;, \tag{1}$$

where we omitted the light-cone indices of odd coordinates. The superscript "n,m" stands for two independent harmonic $U(1)$ charges, left (n) and right (m) ones.

It was argued in [12] that this type of harmonic superspace is most appropriate for constructing off-shell formulations of $(4,4)$ sigma models with torsion. This hope mainly relied upon the fact that the twisted $(4,4)$ multiplet has a natural description as a real analytic $SU(2)\times SU(2)$ harmonic superfield $q^{1,1}(\zeta, u, v)$ (subjected to some harmonic constraints). The most general off-shell action of n such multiplets is given by the following integral over the analytic superspace (1) [12]

$$S_{q,\omega} = \int \mu^{-2,-2}\{q^{1,1\, M}(\; D^{2,0}\omega^{-1,1\, M} + D^{0,2}\omega^{1,-1\, M}\;) + h^{2,2}(q^{1,1}, u, v)\}\;\; (M = 1, ..., n)\;. \tag{2}$$

where

$$\begin{aligned} D^{2,0} &= \partial^{2,0} + i\theta^{1,0\,\underline{i}}\theta^{1,0}_{\underline{i}}\partial_{++}\;,\;\; D^{0,2} = \partial^{0,2} + i\theta^{0,1\,\underline{a}}\theta^{0,1}_{\underline{a}}\partial_{--} \\ (\partial^{2,0} &= u^{1\, i}\frac{\partial}{\partial u^{-1\, i}}\;,\;\; \partial^{0,2} = v^{1\, a}\frac{\partial}{\partial v^{-1\, a}}) \end{aligned} \tag{3}$$

are the left and right analyticity-preserving harmonic derivatives and $\mu^{-2,-2}$ is the analytic superspace integration measure. In (2) the involved superfields are unconstrained analytic, so from the beginning the action (2) contains an infinite number of auxiliary fields coming from the double harmonic expansions with respect to the harmonics $u^{\pm 1\, i}, v^{\pm 1\, a}$. However, after varying with respect to the Lagrange multipliers $\omega^{1,-1\, M}, \omega^{-1,1\, M}$, one comes to the action written only in terms of $q^{1,1\, N}$ subjected to the harmonic constraints

$$D^{2,0}q^{1,1\, M} = D^{0,2}q^{1,1\, M} = 0\;. \tag{4}$$

For each value of M these constraints define the $(4,4)$ twisted multiplet in the $SU(2)\times SU(2)$ harmonic superspace ($8+8$ components off-shell), so the action (2) is a dual form of the general off-shell action of $(4,4)$ twisted multiplets [12].

The crucial feature of the action (2) is the abelian gauge invariance

$$\delta\, \omega^{1,-1\, M} = D^{2,0}\sigma^{-1,-1\, M}\;,\; \delta\, \omega^{-1,1\, M} = -D^{0,2}\sigma^{-1,-1\, M} \tag{5}$$

where $\sigma^{-1,-1\, M}$ are iunconstrained analytic superfield parameters. This gauge freedom ensures the on-shell equivalence of the q,ω formulation of the twisted multiplet action to its original q formulation [12]: it neutralizes superfluous physical dimension component fields in the superfields $\omega^{1,-1\, M}$ and $\omega^{-1,1\, M}$ and thus equalizes the number of propagating fields in both formulations. It holds already at the free level, with $h^{2,2}$ quadratic in $q^{1,1\, M}$, so it is natural to expect that any reasonable generalization of the action (2) respects this symmetry or a generalization of it. We will see soon that this is indeed so.

The dual twisted multiplet action (2) is a good starting point for constructing more general actions which, as we will show, encompass sigma models with non-commuting left and right complex structures.

It is useful to apply to the suggestive analogy with the general action of hyper-Kähler $(4,4)$ sigma models in the $SU(2)$ harmonic superspace [19]. This action in the ω, L^{+2} representation [4] looks very similar to (2), the $SU(2)$ analytic superfield pair $\omega^M, L^{+2\,M}$ being the clear analog of the $SU(2)\times SU(2)$ analytic superfield triple $\omega^{1,-1\,M}, \omega^{-1,1\,M}, q^{1,1\,M}$ and the general hyper-Kähler potential being analogous to $h^{2,2}$. However, this analogy breaks in that the hyper-Kähler potential is in general an arbitrary function of all involved superfields and harmonics while $h^{2,2}$ in (2) depends only on $q^{1,1\,M}$ and harmonics. Thus an obvious way to generalize (2) to cover a wider class of torsionful $(4,4)$ models is to allow for a dependence on $\omega^{1,-1\,M}, \omega^{-1,1\,M}$ in $h^{2,2}$.

With these reasonings in mind, we take as an ansatz for the general action the following one

$$S_{gen} = \int \mu^{-2,-2}\{q^{1,1\,M}(\ D^{2,0}\omega^{-1,1\,M} + D^{0,2}\omega^{1,-1\,M}\ ) + H^{2,2}(q^{1,1}, \omega^{1,-1}, \omega^{-1,1}, u, v)\}\ , \quad (6)$$

where for the moment the ω dependence in $H^{2,2}$ is not fixed. Now we are approaching the most important point. Namely, we are going to show that, contrary to the case of $SU(2)$ harmonic action of torsionless $(4,4)$ sigma models, the ω dependence of the potential $H^{2,2}$ in (6) is completely specified by the integrability conditions following from the commutativity relation

$$[\ D^{2,0}, D^{0,2}\ ] = 0\ . \quad (7)$$

To this end, let us write the equations of motion corresponding to (6)

$$D^{2,0}\omega^{-1,1\,M} + D^{0,2}\omega^{1,-1\,M} = -\frac{\partial H^{2,2}(q,\omega,u,v)}{\partial q^{1,1\,M}}\ , \quad (8)$$

$$D^{2,0}q^{1,1\,M} = \frac{\partial H^{2,2}(q,\omega,u,v)}{\partial \omega^{-1,1\,M}}\ , \quad D^{0,2}q^{1,1\,M} = \frac{\partial H^{2,2}(q,\omega,u,v)}{\partial \omega^{1,-1\,M}}\ . \quad (9)$$

Applying the intgrability condition (7) to the pair of equations (9) and imposing a natural requirement that it is satisfied as a consequence of the equations of motion (i.e. does not give rise to any new dynamical restrictions), after some algebra we arrive at the following set of self-consistency relations

$$\frac{\partial^2 H^{2,2}}{\partial\omega^{-1,1\,N}\partial\omega^{-1,1\,M}} = \frac{\partial^2 H^{2,2}}{\partial\omega^{1,-1\,N}\partial\omega^{1,-1\,M}} = \frac{\partial^2 H^{2,2}}{\partial\omega^{1,-1\,(N}\partial\omega^{-1,1\,M)}} = 0\ , \quad (10)$$

$$\begin{aligned}&\left(\partial^{2,0} + \frac{\partial H^{2,2}}{\partial\omega^{-1,1\,N}}\frac{\partial}{\partial q^{1,1\,N}} - \frac{1}{2}\frac{\partial H^{2,2}}{\partial q^{1,1\,N}}\frac{\partial}{\partial\omega^{-1,1\,N}}\right)\frac{\partial H^{2,2}}{\partial\omega^{1,-1\,M}} \\ &- \left(\partial^{0,2} + \frac{\partial H^{2,2}}{\partial\omega^{1,-1\,N}}\frac{\partial}{\partial q^{1,1\,N}} - \frac{1}{2}\frac{\partial H^{2,2}}{\partial q^{1,1\,N}}\frac{\partial}{\partial\omega^{1,-1\,N}}\right)\frac{\partial H^{2,2}}{\partial\omega^{-1,1\,M}} = 0\ . \quad (11)\end{aligned}$$

Eqs. (10) imply

$$\begin{aligned}H^{2,2} &= h^{2,2}(q,u,v) + \omega^{1,-1\,N}h^{1,3\,N}(q,u,v) + \omega^{-1,1\,N}h^{3,1\,N}(q,u,v) \\ &\quad + \omega^{-1,1\,N}\omega^{1,-1\,M}h^{2,2\,[N,M]}(q,u,v)\ . \quad (12)\end{aligned}$$

Plugging this expression into the constraint (11), we finally deduce four independent constraints on the potentials $h^{2,2}$, $h^{1,3\,N}$, $h^{3,1\,N}$ and $h^{2,2\,[N,M]}$

$$\nabla^{2,0} h^{1,3\,N} - \nabla^{0,2} h^{3,1\,N} + h^{2,2\,[N,M]} \, \frac{\partial h^{2,2}}{\partial q^{1,1\,M}} \;=\; 0 \tag{13}$$

$$\nabla^{2,0} h^{2,2\,[N,M]} - \frac{\partial h^{3,1\,N}}{\partial q^{1,1\,T}} \, h^{2,2\,[T,M]} + \frac{\partial h^{3,1\,M}}{\partial q^{1,1\,T}} \, h^{2,2\,[T,N]} \;=\; 0 \tag{14}$$

$$\nabla^{0,2} h^{2,2\,[N,M]} - \frac{\partial h^{1,3\,N}}{\partial q^{1,1\,T}} \, h^{2,2\,[T,M]} + \frac{\partial h^{1,3\,M}}{\partial q^{1,1\,T}} \, h^{2,2\,[T,N]} \;=\; 0 \tag{15}$$

$$h^{2,2\,[N,T]} \, \frac{\partial h^{2,2\,[M,L]}}{\partial q^{1,1\,T}} + h^{2,2\,[L,T]} \, \frac{\partial h^{2,2\,[N,M]}}{\partial q^{1,1\,T}} + h^{2,2\,[M,T]} \, \frac{\partial h^{2,2\,[L,N]}}{\partial q^{1,1\,T}} \;=\; 0 \tag{16}$$

where

$$\nabla^{2,0} = \partial^{2,0} + h^{3,1\,N} \frac{\partial}{\partial q^{1,1\,N}} \;, \;\; \nabla^{0,2} = \partial^{0,2} + h^{1,3\,N} \frac{\partial}{\partial q^{1,1\,N}} \;. \tag{17}$$

and $\partial^{2,0}, \partial^{0,2}$ act only on the "target" harmonics, i.e. those appearing explicitly in the potentials.

Thus we have shown that the direct generalization of the generic hyper-Kähler $(4,4)$ sigma model action to the torsionful case is given by the action

$$\begin{aligned} S_{q,\omega} \;\; = \;\; & \int \mu^{-2,-2} \{ \, q^{1,1\,M} D^{0,2} \omega^{1,-1\,M} + q^{1,1\,M} D^{2,0} \omega^{-1,1\,M} + \omega^{1,-1\,M} h^{1,3\,M} \\ & + \omega^{-1,1\,M} h^{3,1\,M} + \omega^{-1,1\,M} \omega^{1,-1\,N} \, h^{2,2\,[M,N]} + h^{2,2} \, \} \,, \end{aligned} \tag{18}$$

where the involved potentials depend only on $q^{1,1\,M}$ and target harmonics and satisfy the target space constraints (13) - (16). These constraints certainly encode a nontrivial geometry which for the time being is unclear to us. To reveal it we need to solve the constraints, which is still to be done. At present we are only aware of their particular solution which will be discussed in the next section.

In the rest of this section we present a set of invariances of the action (18) and constraints (13) - (16) which can be useful for understanding the underlying geometry of the given class of sigma models.

One of these invariances is a mixture of reparametrizations in the target space (spanned by the involved superfields and target harmonics) and the transformations which are biharmonic analogs of hyper-Kähler transformations of Refs. [20, 4]. It is realized by

$$\begin{aligned} \delta q^{1,1\,N} \;\; &= \;\; \lambda^{1,1\,N} \;, \;\; \delta\omega^{-1,1\,N} \;=\; -\frac{\partial \lambda^{0,2}}{\partial q^{1,1\,N}} - \frac{\partial \lambda^{1,1\,M}}{\partial q^{1,1\,N}} \, \omega^{-1,1\,M} \;, \\ \delta\omega^{1,-1\,N} \;\; &= \;\; -\frac{\partial \lambda^{2,0}}{\partial q^{1,1\,N}} - \frac{\partial \lambda^{1,1\,M}}{\partial q^{1,1\,N}} \, \omega^{1,-1\,M} \;, \\ \delta h^{2,2} \;\; &= \;\; \nabla^{2,0} \lambda^{0,2} + \nabla^{0,2} \lambda^{2,0} \;, \\ \delta h^{3,1\,M} \;\; &= \;\; \nabla^{2,0} \lambda^{1,1\,M} + h^{2,2\,[M,N]} \, \frac{\partial \lambda^{2,0}}{\partial q^{1,1\,N}} \\ \delta h^{1,3\,M} \;\; &= \;\; \nabla^{0,2} \lambda^{1,1\,M} - h^{2,2\,[M,N]} \, \frac{\partial \lambda^{0,2}}{\partial q^{1,1\,N}} \\ \delta h^{2,2\,[N,M]} \;\; &= \;\; \frac{\partial \lambda^{1,1\,N}}{\partial q^{1,1\,L}} \, h^{2,2\,[L,M]} - \frac{\partial \lambda^{1,1\,M}}{\partial q^{1,1\,L}} \, h^{2,2\,[L,N]} \;, \end{aligned} \tag{19}$$

all the involved transformation parameters being unconstrained functions of $(q^{1,1\,M},u,v)$. This kind of invariance can be used to bring the potentials in (18) into a "normal" form similar to the normal gauge of the hyper-Kähler potential (see [4]).

Much more interesting is another invariance which has no analog in the hyper-Kähler case and is a nonabelain and in general nonlinear generalization of the abelian gauge invariance (5)

$$
\begin{aligned}
\delta\omega^{1,-1\,M} &= \left(D^{2,0}\delta^{MN} + \frac{\partial h^{3,1\,N}}{\partial q^{1,1\,M}}\right)\sigma^{-1,-1\,N} - \omega^{1,-1\,L}\,\frac{\partial h^{2,2\,[L,N]}}{\partial q^{1,1\,M}}\,\sigma^{-1,-1\,N}\;, \\
\delta\omega^{-1,1\,M} &= -\left(D^{0,2}\delta^{MN} + \frac{\partial h^{1,3\,N}}{\partial q^{1,1\,M}}\right)\sigma^{-1,-1\,N} - \omega^{-1,1\,L}\,\frac{\partial h^{2,2\,[L,N]}}{\partial q^{1,1\,M}}\,\sigma^{-1,-1\,N}\;, \\
\delta q^{1,1\,M} &= \sigma^{-1,-1\,N}h^{2,2\,[N,M]}\;. \qquad (20)
\end{aligned}
$$

As expected, the action is invariant only with taking account of the integrability conditions (13) - (16). In general, these gauge transformations close with a field-dependent Lie bracket parameter. Indeed, commuting two such transformations, say, on $q^{1,1\,N}$, and using the cyclic constraint (16), we find

$$
\delta_{br}q^{1,1\,M} = \sigma_{br}^{-1,-1\,N}h^{2,2\,[N,M]}\;,\;\; \sigma_{br}^{-1,-1\,N} = -\sigma_1^{-1,-1\,L}\sigma_2^{-1,-1\,T}\frac{\partial h^{2,2\,[L,T]}}{\partial q^{1,1\,N}}\;. \qquad (21)
$$

We see that eq. (16) guarantees the nonlinear closure of the algebra of gauge transformations (20) and so it is a group condition similar to the Jacobi identities.

Curiously enough, the gauge transformations (20) augmented with the group condition (16) are precise bi-harmonic counterparts of the two-dimensional version of basic relations of the Poisson nonlinear gauge theory which received some attention recently [17, 18] (with the evident correspondence $D^{2,0}, D^{0,2} \leftrightarrow \partial_\mu$; $\omega^{1,-1\,M}, -\omega^{-1,1\,M} \leftrightarrow A^M_\mu$; $\mu = 1,2$). The action (18) coincides in appearance with the general (non-topological) action of Poisson gauge theory [18]. The manifold (q,u,v) can be interpreted as a kind of bi-harmonic extension of some Poisson manifold and the potential $h^{2,2\,[N,M]}(q,u,v)$ as a tensor field inducing the Poisson structure on this extension. We find it remarkable that the harmonic superspace action of torsionful $(4,4)$ sigma models deduced using an analogy with hyper-Kähler $(4,4)$ sigma models proved to be a direct harmonic counterpart of the nonlinear gauge theory action constructed in [17, 18] by entirely different reasoning! We believe that this exciting analogy is a clue to the understanding of the intrinsic geometry of general $(4,4)$ sigma models with torsion.

To avoid a possible confusion, it is worth mentioning that the theory considered *is not* a supersymmetric extension of any genuine $2D$ gauge theory: there are no gauge fields in the multiplet of physical fields. The only role of gauge invariance (20) seems to consist in ensuring the correct number of the sigma model physical fields ($4n$ bosonic and $8n$ fermionic ones).

It should be pointed out that it is the presence of the antisymmetric potential $h^{2,2\,[N,M]}$ that makes the considered case nontrivial and, in particular, the gauge invariance (20) nonabelian. If $h^{2,2\,[N,M]}$ is vanishing, the invariance gets abelian and the constraints (14) - (16) are identically satisfied, while (13) is solved by

$$
h^{1,3\,M} = \nabla^{0,2}\Sigma^{1,1\,M}(q,u,v),\;\; h^{3,1\,M} = \nabla^{2,0}\Sigma^{1,1\,M}(q,u,v)\;, \qquad (22)
$$

with $\Sigma^{1,1\,M}$ being an unconstrained prepotential. Then, using the target space gauge symmetry (19), one may entirely gauge away $h^{1,3\,M}, h^{3,1\,M}$, thereby reducing (18) to the dual action of twisted $(4,4)$ multiplets (2). In the case of one triple $q^{1,1}, \omega^{1,-1}, \omega^{-1,1}$ the potential $h^{2,2\,[N,M]}$ vanishes identically, so the general action (6) for $n = 1$ is actually equivalent to (2). Thus only for $n \geq 2$ a new class of torsionful $(4,4)$ sigma models comes out. It is easy to see that the action (18) with non-zero $h^{2,2\,[N,M]}$ *does not* admit any duality transformation to the form with the superfields $q^{1,1\,M}$ only, because it is impossible to remove the dependence on $\omega^{1,-1\,N}, \omega^{-1,1\,N}$ from the equations for $q^{1,1\,M}$ by any local field redefinition with preserving harmonic analyticity. Moreover, in contradistinction to the constraints (4), these equations are compatible only with using the equation for ω's. So, the obtained system definitely does not admit in general any dual description in terms of twisted $(4,4)$ superfields. Hence, the left and right complex structures on the target space can be non-commuting. In the next section we will explicitly show this non-commutativity for a particular class of the models in question.

3. Harmonic Yang-Mills sigma models. Here we present a particular solution to the constraints (13)-(16). We believe that it shares many features of the general solution which is as yet unknown.

It is given by the following ansatz

$$\begin{aligned} h^{1,3\,N} &= h^{3,1\,N} = 0\;;\; h^{2,2} = h^{2,2}(t,u,v)\;,\;\; t^{2,2} = q^{1,1\,M}q^{1,1\,M}\;; \\ h^{2,2\,[N,M]} &= b^{1,1}f^{NML}q^{1,1\,L}\;,\; b^{1,1} = b^{ia}u^1_i v^1_a\;,\; b^{ia} = \text{const}\;, \end{aligned} \tag{23}$$

where the real constants f^{NML} are totally antisymmetric. The constraints (13) - (15) are identically satisfied with this ansatz, while (16) is now none other than the Jacobi identity which tells us that the constants f^{NML} are structure constants of some real semi-simple Lie algebra (the minimal possibility is $n = 3$, the corresponding algebra being $so(3)$). Thus the $(4,4)$ sigma models associated with the above solution can be interpreted as a kind of Yang-Mills theories in the harmonic superspace. They provide the direct nonabelian generalization of the twisted multiplet sigma models with the action (2) which are thus analogs of two-dimensional abelian gauge theory. The action (18), related equations of motion and the gauge transformation laws (20) specialized to the case (23) are as follows

$$\begin{aligned} S^{YM}_{q,\omega} = \int \mu^{-2,-2}\{\, & q^{1,1\,M}(\,D^{0,2}\omega^{1,-1\,M} + D^{2,0}\omega^{-1,1\,M} + b^{1,1}\,\omega^{-1,1\,L}\omega^{1,-1\,N}f^{LNM}\,) \\ & + h^{2,2}(q,u,v)\} \end{aligned} \tag{24}$$

$$\begin{aligned} D^{2,0}\omega^{-1,1\,N} + D^{0,2}\omega^{1,-1\,N} + b^{1,1}\,\omega^{-1,1\,S}\omega^{1,-1\,T}f^{STN} &\equiv B^{1,1\,N} = -\frac{\partial h^{2,2}}{\partial q^{1,1\,N}}\,, \\ D^{2,0}q^{1,1\,M} + b^{1,1}\,\omega^{1,-1\,N}f^{NML}q^{1,1\,L} &\equiv \Delta^{2,0}q^{1,1\,M} = 0 \\ D^{0,2}q^{1,1\,M} - b^{1,1}\,\omega^{-1,1\,N}f^{NML}q^{1,1\,L} &\equiv \Delta^{0,2}q^{1,1\,M} = 0 \end{aligned} \tag{25}$$

$$\begin{aligned} \delta\omega^{1,-1\,M} &= \Delta^{2,0}\sigma^{-1,-1\,M}\,,\;\; \delta\omega^{-1,1\,M} = -\Delta^{0,2}\sigma^{-1,-1\,M}\,, \\ \delta q^{1,1\,M} &= b^{1,1}\,\sigma^{-1,-1\,N}f^{NML}q^{1,1\,L}\,. \end{aligned} \tag{26}$$

These formulas make the analogy with two-dimensional nonabelian gauge theory almost literal, especially for

$$h^{2,2} = q^{1,1\,M}q^{1,1\,M}\,. \tag{27}$$

Under this choice

$$q^{1,1\,N} = -\frac{1}{2}\, B^{1,1\,N}$$

by first of eqs. (25), then two remaining equations are direct analogs of two-dimensional Yang-Mills equations

$$\Delta^{2,0} B^{1,1\,N} = \Delta^{0,2} B^{1,1\,N} = 0\;, \tag{28}$$

and we recognize (24) and (25) as the harmonic counterpart of the first order formalism of two-dimensional Yang-Mills theory. In the general case $q^{1,1\,M}$ is a nonlinear function of $B^{1,1\,N}$, however for $B^{1,1\,N}$ one still has the same equations (28).

Now it is a simple exercise to see that in checking the integrability condition (7) one necessarily needs first of eqs. (25), while in the abelian, twisted multiplet case this condition is satisfied without any help from the equation obtained by varying the action (2) with respect to $q^{1,1\,N}$. This property reflects the fact that the class of $(4,4)$ sigma models we have found cannot be described only in terms of twisted $(4,4)$ multiplets (of course, in general the gauge group has the structure of a direct product with abelian factors; the relevant $q^{1,1}$'s satisfy the linear twisted multiplet constraints (4)).

An interesting specific feature of this "harmonic Yang-Mills theory" is the presence of the doubly charged "coupling constant" $b^{1,1}$ in all formulas, which is necessary for the correct balance of the harmonic $U(1)$ charges. Since $b^{1,1} = b^{ia}u^1_i v^1_a$, we conclude that in the geometry of the considered class of $(4,4)$ sigma models a very essential role is played by the quartet constant b^{ia}. When $b^{ia} \rightarrow 0$, the nonabelian structure contracts into the abelian one and we reproduce the twisted multiplet action (2). We shall see soon that b^{ia} measures the "strength" of non-commutativity of the left and right complex structures.

Let us limit ourselves to the simplest case (27) and compute the relevant bosonic sigma model action and complex structures. We will do this to the first order in physical bosonic fields, which will be sufficient to show the non-commutativity of complex structures.

We first impose a kind of Wess-Zumino gauge with respect to the local symmetry (26). We choose it so as to gauge away from $\omega^{1,-1\,N}$ as many components as possible, while keeping $\omega^{-1,1\,N}$ and $q^{1,1\,N}$ arbitrary. The gauge-fixed form of $\omega^{1,-1\,N}$ is as follows

$$\omega^{1,-1\,N}(\zeta,u,v) = \theta^{1,0\,\underline{i}}\; \nu^{0,-1\,N}_{\underline{i}}(\zeta_R,v) + \theta^{1,0}\theta^{1,0}\; g^{0,-1\,iN}(\zeta_R,v)u^{-1}_i \tag{29}$$

with

$$\{\zeta_R\} \equiv \{x^{++}, x^{--}, \theta^{0,1\,\underline{a}}\}\;.$$

Then we substitute (29) into (24) with $h^{2,2}$ given by (27), integrate over θ's and u's, eliminate infinite tails of decoupling auxiliary fields and, after this routine work, find the physical bosons part of the action (24) as the following integral over x and harmonics v

$$S_{bos} = \int d^2x[dv]\left(\frac{i}{2}\, g^{0,-1\,iM}(x,v)\;\partial_{--}q^{0,1\,M}_i(x,v)\right)\;. \tag{30}$$

Here the fields g and q are subjected to the harmonic differential equations

$$\begin{aligned}
\partial^{0,2}g^{0,-1\,iM} - 2(b^{ka}v^1_a)\; f^{MNL}q^{0,1\,iN}g^{0,-1\,L}_k &= 4i\partial_{++}q^{0,1\,iM} \\
\partial^{0,2}q^{0,1\,iM} - 2f^{MLN}(b^{ka}v^1_a)\; q^{0,1\,L}_k\; q^{0,1\,iN} &= 0
\end{aligned} \tag{31}$$

and are related to the initial superfields as

$$q^{1,1\,M}(\zeta,u,v)| = q^{0,1\,iM}(x,v)u^1_i + ...\ , \quad g^{0,-1\,iN}(\zeta_R,v)| = g^{0,-1\,iN}(x,v)\ ,$$

where $|$ means restriction to the θ independent parts.

To obtain the ultimate form of the action as an integral over x^{++}, x^{--}, we should solve eqs. (31), substitute the solution into (30) and do the v integration. Here we solve (31) to the first non-vanishing order in the physical bosonic field $q^{ia\,M}(x)$ which appears as the first component in the v expansion of $q^{0,1\,iM}$

$$q^{0,1\,iM}(x,v) = q^{ia\,M}(x)v^1_a + ...\ .$$

Representing (30) as

$$S_{bos} = \int d^2x \left(G^{M\ L}_{ia\ kb}\partial_{++}q^{ia\,M}\partial_{--}q^{kb\,L} + B^{M\ L}_{ia\ kb}\partial_{++}q^{ia\,M}\partial_{--}q^{kb\,L}\right) \tag{32}$$

where the metric G and the torsion potential B are, respectively, symmetric and skew-symmetric with respect to the simultaneous interchange of the left and right triples of their indices, we find that to the first order

$$G^{M\ L}_{ia\ kb} = \delta^{ML}\epsilon_{ik}\epsilon_{ab} - \frac{2}{3}\epsilon_{ik}f^{MLN}b_{l(a}q^{l\,N}_{b)}\ ,\quad B^{M\ L}_{ia\ kb} = \frac{2}{3}f^{MLN}[b_{(ia}q^{N}_{k)b} + b_{(ib}q^{N}_{k)a}]\ . \tag{33}$$

Note that an asymmetry between the indices ik and ab in the metric is an artefact of our choice of the WZ gauge in the form (29). One could choose another gauge so that a symmetry between the above pairs of $SU(2)$ indices is restored. Metrics in different gauges are related via the target space $q^{ia\,M}$ reparametrizations.

Finally, let us compute, to the first order in $q^{ia\,M}$, the left and right complex structures associated with the sigma models at hand. Following the well-known strategy [7, 15], we need: (i) to partially go on shell by eliminating the auxiliary fermionic fields; (ii) to divide four supersymmetries in every light-cone sector into a $N = 1$ one realized linearly and a triplet of nonlinearly realized extra supersymmetries; (iii) to consider the transformation laws of the physical bosonic fields $q^{ia\,M}$ under extra supersymmetries. The complex structures can be read off from these transformation laws.

In our case at the step (i) we should solve some harmonic differential equations of motion to express an infinite tail of auxiliary fermionic fields in terms of the physical ones and the bosonic fields $q^{ia\,M}$. The step (ii) amounts to the decomposition of the $(4,0)$ and $(0,4)$ supersymmetry parameters $\varepsilon^{i\underline{i}}_-$ and $\varepsilon^{a\underline{a}}_+$ as

$$\varepsilon^{i\underline{i}\,+} \equiv \epsilon^{i\underline{i}}\varepsilon^{+} + i\varepsilon^{(i\underline{i})\,+}\ ,\quad \varepsilon^{a\underline{a}\,-} \equiv \epsilon^{a\underline{a}}\varepsilon^{-} + i\varepsilon^{(a\underline{a})\,-}\ ,$$

where we have kept a manifest symmetry only with respect to the diagonal $SU(2)$ groups in the full left and right automorphism groups $SO(4)_L$ and $SO(4)_R$. At the step (iii) we should redefine the physical fermionic fields so that the singlet supersymmetries with the parameters ε_- and ε_+ be realized linearly. We skip the details and present the final form of the on-shell supersymmetry transformations of $q^{ia\,M}(x)$

$$\delta q^{ia\,M} = \varepsilon^{+}\,\psi^{ia\,M}_{+} + i\varepsilon^{(kj)\,+}\,\left(F_{(kj)}\right)^{ia\,M}_{lb\,L}\,\psi^{lb\,L}_{+} + \varepsilon^{-}\,\chi^{ia\,M}_{-} + i\varepsilon^{(cd)\,-}\,\left(\hat{F}_{(cd)}\right)^{ia\,M}_{lb\,L}\,\chi^{lb\,L}_{-}\ . \tag{34}$$

Introducing the matrices

$$F^n_{(+)} \equiv (\tau^n)^k_j F^{(j}{}_{k)}\ , \quad F^m_{(-)} \equiv (\tau^m)^c_d \hat{F}^{(d}{}_{c)}\ ,$$

τ^n being Pauli matrices, we find that in the first order in $q^{ia\,M}$ and b^{ia}

$$\begin{aligned} F^n_{(+)} &= -i\tau^n \otimes I \otimes I + \frac{i}{3}\,[\,M_{(+)}, \tau^n \otimes I \otimes I\,] \\ F^n_{(-)} &= -iI \otimes \tau^n \otimes I + \frac{i}{3}\,[\,M_{(-)}, I \otimes \tau^n \otimes I\,] \end{aligned} \tag{35}$$

$$\left(M_{(+)}\right)^{ia\,M}_{kb\,N} = -2\,f^{MLN}\left(b^{(i}_{b}q^{a\,L}{}_{k)} + b^{(ia}q^{L}{}_{k)b}\right)\ , \quad \left(M_{(-)}\right)^{ia\,M}_{kb\,N} = 2\,f^{MLN}\,b^{i}_{(b}q^{a)\,L}_{k}\ , \tag{36}$$

where the matrix factors in the tensor products are arranged so that they act, respectively, on the subsets of indices $i, j, k, ...,\ a, b, c, ...,\ M, N, L, ...$.

It is easy to check that the matrices $F^n_{(\pm)}$ to the first order in q, b possess all the standard properties of complex structures [7]. In particular, they form a quaternionic algebra

$$F^n_{(\pm)}F^m_{(\pm)} = -\delta^{nm} + \epsilon^{nms}F^s_{(\pm)}\ ,$$

and satisfy the covariant constancy conditions

$$\mathcal{D}_{lc\,K}\left(F^n_{(\pm)}\right)^{ia\,M}_{kb\,N} = \partial_{lc\,K}\left(F^n_{(\pm)}\right)^{ia\,M}_{kb\,N} - \Gamma_{(\pm)\,lc\,K}{}^{jd\,T}{}_{kb\,N}\left(F^n_{(\pm)}\right)^{ia\,M}_{jd\,T} + \Gamma_{(\pm)\,lc\,K}{}^{ia\,M}{}_{jd\,T}\left(F^n_{(\pm)}\right)^{jd\,T}_{kb\,N} = 0$$

with

$$\Gamma_{(\pm)\,lc\,M}{}^{jd\,T}{}_{kb\,N} \equiv \Gamma_{lc\,M}{}^{jd\,T}{}_{kb\,N} \mp T_{lc\,M}{}^{jd\,T}{}_{kb\,N}\ ,$$

where Γ is the standard Riemann connection for the metric (33) and T is the torsion

$$T_{ia\,M\ \ kb\,N\ \ ld\,T} = \frac{1}{2}\left(\partial_{ia\,M}B^{N\,T}_{kb\,ld} + \ \ cyclic\right)\ .$$

Of course, in the present case these properties are fulfilled automatically as we started with a manifestly $(4,4)$ supersymmetric off-shell superfield formulation.

It remains to find the commutator of complex structures. The straightforward computation (again, to the first order in fields) yields

$$\begin{aligned} [\,F^n_{(+)}, F^m_{(-)}\,] &= (\tau^n \otimes I \otimes I)M_{(-)}(I \otimes \tau^m \otimes I) + (I \otimes \tau^m \otimes I)M_{(-)}(\tau^n \otimes I \otimes I) \\ &- (\tau^n \otimes \tau^m \otimes I)M_{(-)} - M_{(-)}(\tau^n \otimes \tau^m \otimes I)\ \neq\ 0\ . \end{aligned} \tag{37}$$

Thus in the present case in the bosonic sector we encounter a more general geometry compared to the one discussed in [7, 6]. The basic characteristic feature of this new geometry is the non-commutativity of the left and right complex structures. It is easy to check this property also for more general potentials $h^{2,2}(q,u,v)$ in (24). It seems obvious that the general case (18), (13) - (16) reveals the same feature. Stress once more that this important property is related in a puzzling way to the nonabelian structure of the analytic superspace actions (24), (18): the "coupling constant" $b^{1,1}$ (or the Poisson potential $h^{2,2\,[M,N]}$ in the general case) measures the strength of the non-commutativity of complex structures.

4. Conclusion. To summarize, proceeding from an analogy with the $SU(2)$ harmonic superspace description of $(4,4)$ hyper-Kähler sigma models, we have constructed off-shell $SU(2) \times SU(2)$ harmonic superspace actions for a new wide class of $(4,4)$ sigma models with torsion and non-commuting left and right complex structures on the bosonic target. This non-commutativity is directly related to the remarkable non-abelian Poisson gauge structure of these actions. One of the most characteristic features of the general action is the presence of an infinite number of auxiliary fields and the lacking of dual-equivalent formulations in terms of $(4,4)$ superfields with finite sets of auxiliary fields. It would be interesting to see whether such formulations exist for some particular cases, e.g., those corresponding to the bosonic manifolds with isometries. An example of $(4,4)$ sigma model with non-commuting structures which admits such a formulation has been given in [16].

The obvious problems for further study are to compute the relevant metrics and torsions in a closed form and to try to utilize the corresponding manifolds as backgrounds for some superstrings. An interesting question is as to whether the constraints (13) - (16) admit solutions corresponding to the $(4,4)$ supersymmetric group manifold WZNW sigma models. The list of appropriate group manifolds has been given in [8]. The lowest dimension manifold with non-commuting left and right structures [10] is that of $SU(3)$. Its dimension 8 coincides with the minimal bosonic manifold dimension at which a non-trivial $h^{2,2\,[M,N]}$ in (18) can appear.

Of course, it remains to prove that the action (18) indeed describes most general $(4,4)$ models with torsion. One way to do this is to start, like in the hyper-Kähler and quaternionic cases [4, 5], with the constrained formulation of the relevant geometry in a real $4n$ dimensional manifold and to reproduce the potentials in (18) as some fundamental objects which solve the initial constraints. But even before performing such an analysis, there arises the question of how general our original ansatz (6) for the action is. In fact, in [21] we have chosen a more general ansatz and shown that at least for four-dimensional bosonic manifolds it is effectively reduced to (6) and hence to (18) after the combined use of the target space diffeomorphisms and the intregrability condition (7). For general manifolds of the dimension $4n$, $n \geq 2$, this still needs to be checked, in [21] only some heuristic arguments in favour of this have been adduced. Another point is that the constrained superfield $q^{1,1\,M}$ the dual action of which was a starting point of our construction, actually comprises only one type of $(4,4)$ twisted multiplet [13]. There exist other types which differ in the $SU(2)_L \times SU(2)_R$ assignment of their components [7, 11]. At present it is unclear how to simultaneously describe all of them in the framework of the $SU(2) \times SU(2)$ analytic harmonic superspace. Perhaps, their actions are related to those of $q^{1,1}$ by a kind of duality transformation. It may happen, however, that for their self-consistent description one will need a more general type of $(4,4)$ harmonic superspace, with the whole $SO(4)_L \times SO(4)_R$ automorphism group of $(4,4)$ SUSY harmonized. The relevant actions will be certainly more general than those considered in this paper.

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